Physics 54

Steady Currents

I've had a lovely evening — but this wasn't it.
— Groucho Marx

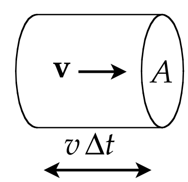
Electric current

One might think the discussion of charges in motion would begin with a description of the fields of a single moving point charge. But those fields are quite complicated — even if the charge is moving with constant velocity, but especially if it is accelerating.

However, when we consider the average fields produced by a large number of identical charges moving (on average) relatively slowly, the situation is much simpler . Especially simple is the case where the charges move with constant average velocity. In that case, called **direct current** (DC), the flow of charge is much like that of mass in a perfect fluid.

To describe the flow we introduce the **electric current density**:

Current density	Electric current density $\mathbf{j}$ is a vector field. Its direction is that of the flow of positive charge; its magnitude is equal to the amount of charge passing in unit time through unit area normal to the flow.



To relate the net flow described by $\mathbf{j}$ to the average motion of the individual microscopic charges (such as electrons), consider a small cylinder of cross-section area A and length $v\,\Delta t$, where v is the average speed of the moving charges; the cylinder has volume $Av\,\Delta t$. We assume the moving charges are identical, that each has charge q, and that there are n charges per unit volume.

In time Δt all the charge in the cylinder passes through the area A at the right end; this amount of charge is $nq(Av\,\Delta t)$. So the amount of charge flowing per unit area per unit time is nqv, and we find that

$$\mathbf{j} = nq\mathbf{v}\,.$$

If q is positive, $\mathbf{j}$ is parallel to the velocity $\mathbf{v}$. (For electron flow, $\mathbf{j}$ is opposite to $\mathbf{v}$.)

The flux of **j** through a particular surface gives the rate of flow of charge through that surface. It is called the **electric current**, denoted by I.

Electric current as flux of current density	$I = \int \mathbf{j} \cdot d\mathbf{A}$

In our applications the conductors will usually be in the form of wires, for which the area in question is the cross-section of the wire. The current I is then the total amount of charge passing a particular point on the wire per unit time, which one often writes as

Electric current as charge passing by per unit time	$I = \frac{dQ}{dt}$

Electric current is measured in amperes (A). A current of 1 A means that 1 C of charge passes a point on the wire per second. This is a current of moderate size.

The SI system of electrical units is in fact based on the ampere. One coulomb of charge is defined as the amount flowing in one second past a point in a wire carrying a current of one ampere.

Resistance

We have seen that charges in a conductor, left to themselves, quickly rearrange and come to rest in electrostatic equilibrium. To keep them moving requires continuous application of an external E-field. We will discuss below some possible sources of this external field. For now we will just assume such a field exists.

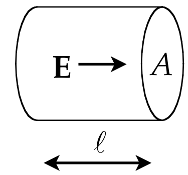
For currents in ordinary conductors (not superconductors) a steady state is quickly reached, in which the energy given to the conduction electrons by the external E-field exactly balances the energy they lose in collisions with the "lattice" of atoms of which the material is made. This lost kinetic energy is manifested in the form of thermal energy of random motion of the lattice particles, a process called **Joule heating**.

In this steady state there remains a small average velocity of the conduction electrons, called the "drift" velocity. This velocity (which is denoted by **v** in the discussion above) is directed opposite to **E** (since electrons have negative charge). The current density **j** is parallel to **E**, and the relation between these vectors defines the **resistivity** of the material, a positive scalar denoted by ρ:

Resistivity	$\mathbf{E} = \rho\mathbf{j}$

For many common conducting materials and E-fields that are not too strong, ρ is approximately independent of E. This property is called **Ohm's Law**. Materials for which it is a good approximation are called "ohmic".

This empirical rule was discovered by Ohm in 1826, and was a significant finding, showing that the current in a circuit is proportional to the battery's "tension," as potential difference was called then.



In many practical devices, the conductors are wires of constant cross-section. Consider a small section of such a wire, as shown. We assume that **j** is uniform across the area *A*, so that $I = jA$. The potential *drop* in the direction of the current (which is also the direction of **E**) is given by $\Delta V = E\ell$. We find, using $E = \rho j$ and $j = I / A$:

$$\Delta V = \frac{\rho \ell}{A} I \, .$$

The factor multiplying *I* in this equation is called the **resistance**. Its definition is:

Resistance	If steady current *I* passes through a passive element, resulting in potential drop ΔV between its ends, then the element has resistance $R = \Delta V / I$.

A "passive" element is one that does not transfer energy into or out of the circuit except by means of its resistance; this excludes "active" elements such as batteries.

In this context, Ohm's Law is a statement that *R* is independent of ΔV and *I*.

Along the way, we have derived a useful formula for the resistance of a cylindrical conductor:

$$R = \frac{\rho \ell}{A} \, .$$

Joule heating

For any element through which current *I* passes, the amount of charge going in one end and out the other in time *dt* is given by $dQ = I\,dt$. If the potential *drop* between the ends of the element is ΔV, then as this bit of charge passes through the element the system *loses* an amount of electrostatic potential energy $dU = \Delta V \cdot dQ$, which is transformed into other kinds of energy. The rate at which this transformation occurs is given by $P = dU / dt = \Delta V \cdot dQ / dt = I \, \Delta V$.

Since the current leaving the element is the same as that entering, the drift speed of the charges does not change: there is no change in *kinetic* energy of their motion. The lost electrostatic *potential* energy is changed by the circuit element to other forms, or transferred to other systems. There are many possibilities, some of which we will discuss as we go along.

Regardless of where the energy goes, we have derived a simple and general formula for the amount of power *delivered to* the element:

Power input to circuit element	$P = I\,\Delta V$

It is important that ΔV here represents the *drop* in potential as the current passes through the element.

In the case of an ordinary wire, in which the potential drop comes only from resistance, the energy input to the wire is converted into thermal energy by Joule heating. In this case, $\Delta V = IR$, so we have a simple formula for that process:

Joule heating	$P = I^2 R$

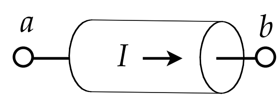
There are many practical applications of this conversion of electric energy into heat, such as incandescent light bulbs and electric space heaters.

EMF and the circuit equation

To maintain a steady current, there must be a supply of energy to replace that lost in Joule heating or in other energy transformations. The source of this energy maintains the E-field from which the charges draw their energy. Such a source is called a "seat of electromotive force" or simply an **emf**. Its energy can arise from many processes.

The strength of an emf is measured by the energy it provides per unit charge, so it is given in volts.

Consider an element through which current I flows. At the end where the current enters let the potential be V_a and where the current exits let it be V_b. We consider the total work done on a bit of charge $dQ = I\,dt$ as it passes through this element.



There are three kinds of work done:

1. Work done *by* the electrostatic field, which is minus the change in electrostatic potential energy: $dW_{elec} = -dQ(V_b - V_a)$.

2. Work (negative) done by the dissipative forces represented by resistance. This is given by the Joule heating formula: $dW_{res} = -I^2 R\,dt = -IR\,dQ$.

3. Work done by all other forces, which we will simply call dW_{other}.

The total work by *all* forces is equal (by the work-energy theorem) to the change in kinetic energy of the charge dQ. But the current is steady, so the kinetic energy does *not* change, which means the total work must add up to zero. We have therefore

$$dW_{other} = (V_b - V_a + IR)\,dQ\,.$$

We define the emf, denoted by $\mathcal{E}$, to be the work done by "other" forces per unit charge:

$$\text{Definition of emf: } \mathcal{E} = \frac{dW_{other}}{dQ}.$$

Substituting from this for dW_{other} and dividing by dQ, we find an important equation:

Circuit equation	$V_b - V_a = \mathcal{E} - IR$

This formula gives the potential difference between any two points in a circuit in terms of the (total) emf and the (total) "IR drop" between the points. It is the basis of one of the fundamental rules for analyzing circuits.

What kinds of things give rise to the "other" forces that produce an emf? The most familiar is a battery, in which the "other" forces arise from chemical reactions, converting chemical energy (the energy binding atoms in molecules) into electric field energy. Various kinds of generators (such as the alternator in a car) convert mechanical energy of some kind—the engine in the car, falling water or steam driving a turbine, etc.—into electrical energy, often by use of Faraday's law, to be discussed later. Photocells of various kinds convert radiation into electric field energy. In all these devices, energy of a different sort is converted to electric field energy, producing an emf to sustain a current.

In our applications in this section, the emf will usually be that of a battery.

DC circuits

A circuit is a system of elements connected by conductors in such a way that currents make complete round trips. In the process, electrical energy is brought in from emfs and transferred by means of the circuit to a "load" where it is converted into other forms, or else passed on to other circuits.

Here we analyze only circuits with currents that do not vary with time (DC). The sources of energy (emfs) will be batteries. The loads will be simple resistances.

The two fundamental principles governing the circuit are:

- Energy balance, expressed by the circuit equation given above.
- Conservation of charge.

These are usually given in the form proposed by Kirchhoff:

Kirchhoff's rules	**Loop Rule.** Over any closed path in a circuit, the total emf of the elements is equal to the total potential drop in the resistances. **Junction Rule.** The total current entering any junction is equal to the total current leaving it.

One applies the loop rule to an arbitrarily chosen *closed* path along the wires of the circuit. Because the starting and ending points are the same, the potential difference is zero, which means that the potential rises in the emf's must be canceled by the potential drops in the resistors.

The junction rule merely states that nowhere does charge simply appear or disappear.

In applying the loop rule, signs are important:

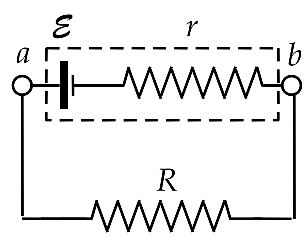
- As an emf is crossed, it is counted as *positive* if the potential *increases*.
- As a resistance is crossed, the "IR drop" is counted as *positive* if the potential actually *drops*, which will happen if chosen path is following the direction of the current.
- One makes assumptions initially about the directions of the currents. If the value of a current turns out by calculation to be negative, it means the assumption was wrong, and that current actually goes the other way (but with the same magnitude).

We will illustrate these rules by some simple examples.

Terminal voltage of a battery

A realistic energy source such as a battery dissipates some energy as current passes through it; we say it has "internal" resistance. As a result, when the current of a circuit passes through this kind of source, the potential difference between its "terminals" (the points where it is connected into the circuit) is generally different from its emf.

In the circuit shown the actual battery is indicated by the dotted rectangle; its terminals are points a and b. The internal resistance is indicated by r.



In circuit diagrams, batteries are indicated by two parallel lines; the longer and heavier line represents the positive (higher potential) side of the battery. Resistors are indicated by zig-zag lines.

When connected to the load resistor R, the battery sustains a current I running counter-clockwise in the circuit. By the loop rule we have

$$I = \frac{\mathcal{E}}{R + r}\,.$$

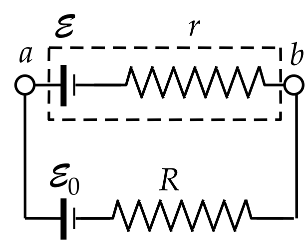
The circuit equation then gives for the terminal voltage

$$V_a - V_b = \mathcal{E} - Ir = \mathcal{E}\frac{R}{R+r}.$$

This is *less* than the emf of the battery. The power delivered to the load resistor is

$$P_R = I^2 R = \mathcal{E}^2 \frac{R}{(R+r)^2}.$$

Treated as a function of *R*, this quantity is a maximum when $R = r$, i.e., when the load resistance is equal to the "source" resistance. This is a simple example of what is called the "maximum power transfer rule."



The next circuit shows a battery (like the one in a car) being charged by an external emf $\mathcal{E}_0$ (such as that of the car's generator).

Here the generator has larger emf than the battery being charged, so the current runs *clockwise*. The current is

$$I = \frac{\mathcal{E}_0 - \mathcal{E}}{R+r}$$

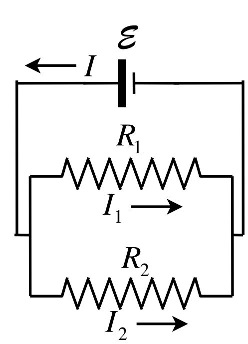
and the terminal voltage of the battery is now

$$V_a - V_b = \mathcal{E} + Ir$$

which is *greater* than the battery emf.

Multiloop circuits

So far we have not needed the junction rule because there are no junctions in a circuit with only one loop.



Consider the circuit shown. We apply the loop rule to the loop passing through the battery and R_1. This gives

$$\mathcal{E} = I_1 R_1.$$

The loop passing through the battery and R_2 gives

$$\mathcal{E} = I_2 R_2.$$

The junction rule gives the final equation:

$$I = I_1 + I_2.$$

These give three equations for the three unknown currents. (The equation obtained from the loop passing only through the two resistors gives nothing new, since it is the difference between the first two equations above.)

Solving for I we find

$$I = \mathcal{E}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

This shows that the two resistors, which are connected in **parallel**, could be replaced by a single effective resistor between points a and b, with resistance R_{eff} given by

$$\text{Parallel: } \frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2}.$$

The current divides as it passes through these resistors, with

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}.$$

The larger current passes through the smaller resistance.

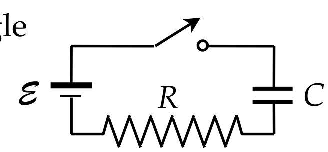
Resistors in **series**, with the same current passing through both, can also be replaced by an effective single resistor. The effective resistance is given by

$$\text{Series: } R_{eff} = R_1 + R_2.$$

R-C circuits

A battery or other source of emf is often used to put charge on a capacitor, transferring energy to its E-field. This transfer does not happen instantly, of course, so the current (representing charge flowing to and from the capacitor plates) varies with time. If the capacitor is initially uncharged, the current is large at first, but decreases as the increasing potential difference across the capacitor impedes it.

We will examine this process in the case of a single battery, a single resistor and a capacitor. The circuit is as shown. The switch is closed at $t = 0$, before which the capacitor had zero charge.



As we traverse the whole circuit, following the current and returning to the same point, the total change in potential is zero. This means that

$$\mathcal{E} - Q/C - IR = 0.$$

Here I is also the rate at which charge is moving *to* the capacitor, so $I = dQ/dt$.

Substituting this and rearranging the equation, we have

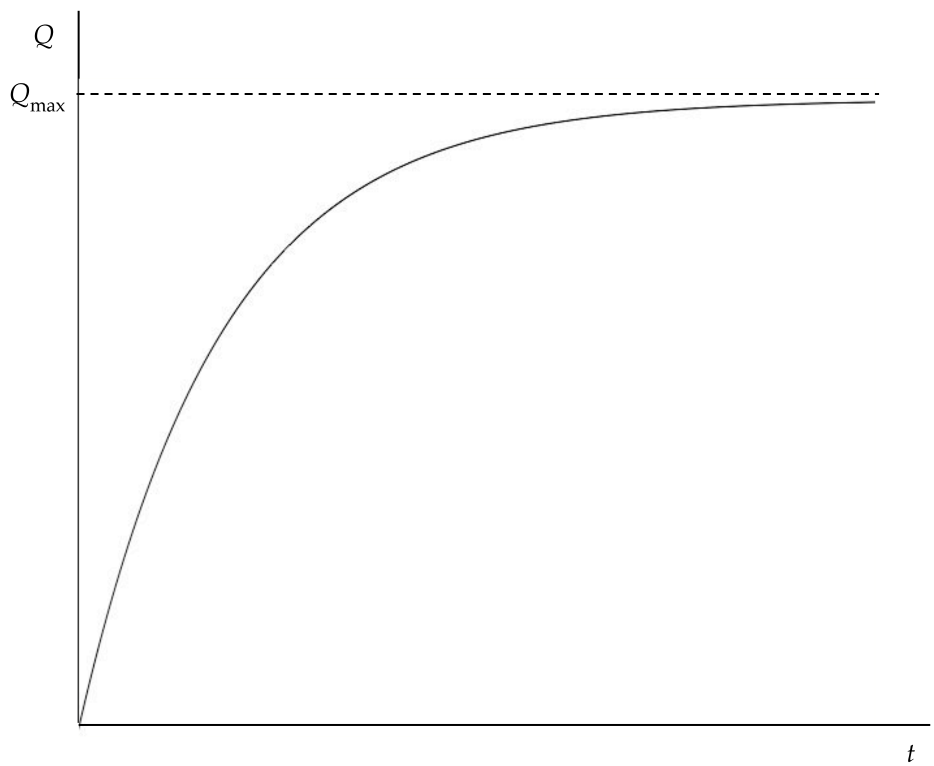
$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{\mathcal{E}}{R}.$$

This is a differential equation to be solved for $Q(t)$. The solution that satisfies the condition that the initial charge be zero is

$$Q(t) = \mathcal{E}C\left(1 - e^{-t/RC}\right).$$

This shows several characteristic properties of the process:

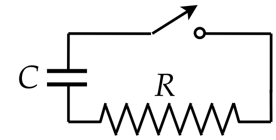
- The maximum charge on the capacitor is $\mathcal{E}C$, corresponding to capacitor voltage equal to $\mathcal{E}$. In this case the capacitor is "fully" charged.
- The charge follows a "saturation" curve as shown in the graph, approaching the maximum charge asymptotically as $t \to \infty$.
- The rate at which the capacitor charge rises is governed by the "time constant" $\tau = RC$. The smaller R the faster the capacitor charges. At time $t = \tau$ the capacitor is charged to within $1/e$ of its maximum.



The current is given by

$$I(t) = dQ/dt = (\mathcal{E}/R)e^{-t/RC}.$$

This exponential decay of the current is also governed by the time constant RC.



In the next circuit shown, a capacitor initially has charge Q_0 and then (at $t = 0$) the switch is closed, connecting the capacitor across the resistor. Charge flows from the positive plate, through the resistor, to the negative plate, eventually discharging the capacitor. We ask how this process depends on time.

In this case, the loop rule gives

$$\frac{Q}{C} - IR = 0,$$

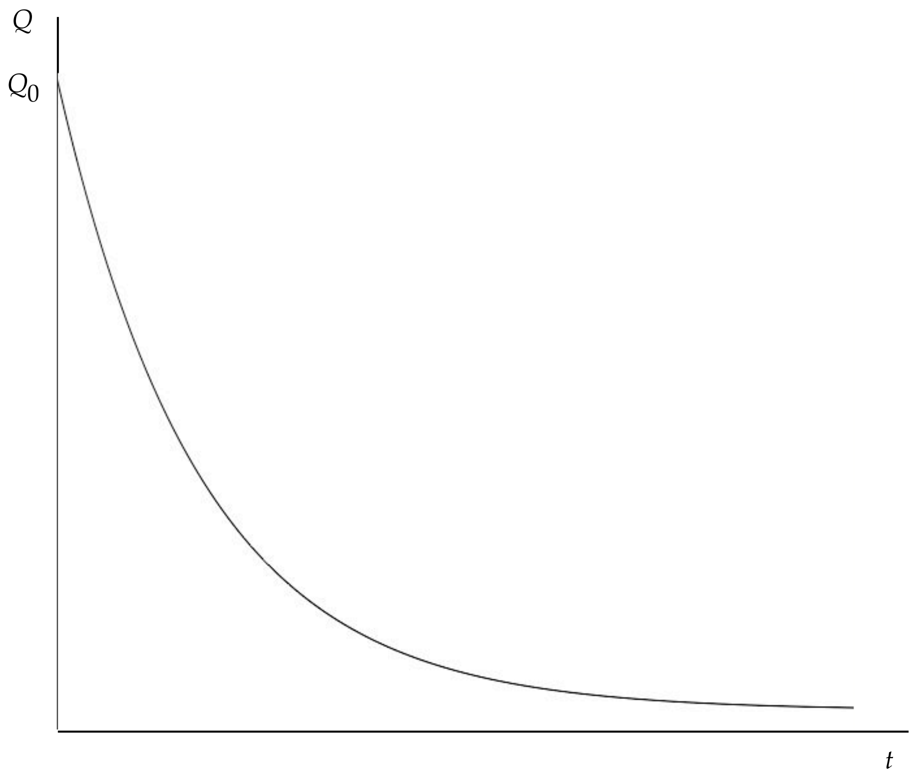
where now (because the capacitor is *discharging*) $I = -dQ/dt$, so we have

$$\frac{dQ}{dt} + \frac{1}{RC}Q = 0.$$

The solution to this differential equation that satisfies the initial condition $Q(0) = Q_0$ is

$$Q(t) = Q_0 e^{-t/RC}.$$

The charge decays exponentially at a rate governed by $\tau = RC$. When $t = \tau$ the charge has dropped to $1/e$ of its initial value. This behavior is shown in the graph.



There are many uses for the charging-discharging process of a capacitor, e.g., in flash photography and defibrillators. Later we will discuss capacitors in AC circuits.